Exercise 19

The rate of change of atmospheric pressure P with respect to altitude h is proportional to P, provided that the temperature is constant. At 15°C the pressure is 101.3 kPa at sea level and 87.14 kPa at $h = 1000$ m.

(a) What is the pressure at an altitude of 3000 m?

(b) What is the pressure at the top of Mount McKinley, at an altitude of 6187 m?

Solution

Assume that the rate of change of atmospheric pressure P with respect to altitude h is proportional to P, provided that the temperature is constant.

$$\frac{dP}{dh} \propto -P$$

The minus sign is included so that as the height increases, the rate of change of pressure is negative. In other words, as the height increases the pressure decreases and vice-versa. Change this proportionality to an equation by introducing a positive constant k.

$$\frac{dP}{dh} = -kP$$

Divide both sides by P.

$$\frac{1}{P}\frac{dP}{dh} = -k$$

Rewrite the left side by using the chain rule.

$$\frac{d}{dh}\ln P = -k$$

The function you take a derivative of to get $-k$ is $-kt + C$, where C is any constant.

$$\ln P = -kh + C$$

Exponentiate both sides to get P.

$$e^{\ln P} = e^{-kh+C}$$

$$P = e^C e^{-kh}$$

Use a different constant P_0 for e^C.

$$P(h) = P_0 e^{-kh}$$

Use the two given pieces of information to constuct a system of equations for P_0 and k, that is, that the pressure is 101.3 kPa at sea level and 87.14 kPa at $h = 1000$ m.

$$\begin{cases} 101.3 = P_0 e^{-k(0)} \\ 87.14 = P_0 e^{-k(1000)} \end{cases}$$

Solve the first equation for P_0.

$$101.3 = P_0 e^{-k(0)} \quad \rightarrow \quad P_0 = 101.3$$

Then plug the result into the second equation and solve for k.

$$87.14 = (101.3)e^{-k(1000)}$$

$$\frac{87.14}{101.3} = e^{-1000k}$$

$$\ln \frac{87.14}{101.3} = \ln e^{-1000k}$$

$$\ln \frac{87.14}{101.3} = (-1000k) \ln e$$

$$k = -\frac{\ln \frac{87.14}{101.3}}{1000} \approx 0.00015057 \text{ meter}^{-1}$$

Consequently, the pressure at height h is

$$\begin{aligned} P(h) &= P_0 e^{-kh} \\ &= 101.3e^{-\left(-\frac{\ln \frac{87.14}{101.3}}{1000}\right)h} \\ &= 101.3e^{\ln\left(\frac{87.14}{101.3}\right)^{h/1000}} \\ &= 101.3\left(\frac{87.14}{101.3}\right)^{h/1000}. \end{aligned}$$

Part (a)

To find the pressure at an altitude of 3000 m, plug in $h = 3000$.

$$P(3000) = 101.3\left(\frac{87.14}{101.3}\right)^{3000/1000} \approx 64.4813 \text{ kPa}$$

Part (b)

To find the pressure at an altitude of 6187 m, plug in $h = 6187$.

$$P(6187) = 101.3\left(\frac{87.14}{101.3}\right)^{6187/1000} \approx 39.9052 \text{ kPa}$$